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On Square-Integrable Representations of A Lie Group of 4 Dimensional Standard Filiform Lie Algebra

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ABSTRACT

In this paper, we study irreducible unitary representations of a real standard filiform Lie group with dimension equals 4 with respect to its basis. To find this representation we apply the orbit method introduced by Kirillov. The corresponding orbit of this representation is generic orbits of dimension 2. Furthermore, we show that obtained representation of this group is square-integrable. Moreover, in such case , we shall consider its Duflo-Moore operator as multiple of scalar identity operator. In our case that scalar is equal to one.

Keywords: Duflo-Moore operator; irreducible unitary representation; square-integrable representation, standard filiform Lie algebra.

INTRODUCTION

In this paper, all Lie algebras are determined over $\mathbb{R}$. The notion of filiform Lie algebras can be found in [1]. In this work, we can find the classification of filiform Lie algebras of dimension ≤ 8 and the computation of their index. Roughly speaking, a Lie algebra $\mathfrak{g}$ of dimension n is said to be filiform if there exists a smallest positive integer ς that equals $n-1$ such that $\mathfrak{g}^{\varsigma} = \{0\}$. We recall that $\mathfrak{g}^n$ is defined as follows : $\mathfrak{g}^1 = \mathfrak{g}, \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$. Particularly, we shall work on a Lie group G of a 4-dimensional standard filiform Lie algebra $\mathfrak{g}$. We determine the irreducible unitary representations of G by applying the orbit method.

The orbit method that shall be applied in this paper comes from [2] and [3]. To bring it down to earth of this method, we can enjoy some examples in [4]. We review this method from [2] and [4] as follows.

Definition 1 [5]. *Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Let g be element of G. We define the conjugation map given by $C_g \colon G \ni x \mapsto gxg^{-1} \in G$ which is a Lie group homomorphism whose differential of C_g is denoted by $\mathrm{Ad}(g) \coloneqq \left(C_g\right)_*$. The adjoint representations of G is given as group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.*

The infinitesimal adjoint representation of Ad is denoted by ad. This map is given by $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ which is defined by $\mathrm{ad}(x)y = [x, y]$ for each $x, y \in \mathfrak{g}$. Moreover, the representations Ad can be defined as a dual representation on a dual vector space $\mathfrak{g}^*$. Namely, we have

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$$\langle \mathrm{Ad}^*(g)f, X\rangle = \langle f, \mathrm{Ad}(\mathrm{g}^{-1})\mathrm{X}\rangle, \quad (\mathrm{g} \in G, f \in \mathfrak{g}^*, X \in \mathfrak{g}) \qquad (1)$$

The coadjoint orbit of $f \in \mathfrak{g}^*$ is defined as a set $\{\mathrm{Ad}^*(g)f \quad ; g \in G\} \subset \mathfrak{g}^*$.

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}^*$ be a dual vector space of $\mathfrak{g}$. To obtain the formula of irreducible unitary representation of G we should consider some steps as follows.

1. For orbit $\Omega \subset \mathfrak{g}^*$, we choose a point $f \in \Omega$.
2. We consider a polarization Υ. Υ is said to be a polarization of $\mathfrak{g}$ if Υ is a subalgebra of $\mathfrak{g}$ and it has maximal dimension, namely its codimension equals $\frac{1}{2}\dim\Omega$, which is subordinate to f. This means $\langle f, [\Upsilon,\Upsilon]\rangle = 0$.
3. Let $\mathrm{H} := \exp \Upsilon$ be a subgroup and let ψ_f be a one dimensional irreducible unitary representations of H given by
$$\psi_f(\exp X) = e^{2\pi i\langle f, X\rangle}, \quad (f \in \Omega, X \in \mathfrak{g}). \qquad (2)$$
4. Identifying the coset space G/H by M and considering a section
$$s\colon \mathrm{M} \to \mathrm{G}. \qquad (3)$$
5. Solving the master equation
$$s(x)g = h(x,g)s(x.g), \quad (x \in \mathrm{M}, g \in G, h(x,g) \in \mathrm{H}), \qquad (4)$$
which is called a **master equation**, and computing the measure on M, we have
$$\pi(g)f(x) = \psi_f\big(h(x,g)\big)f(x.g). \qquad (5)$$

The notion of square-integrable representations can be found in ([6], [7], [8], and [9]). Let $(\pi, \varkappa_\pi)$ be a irreducible unitary representations locally compact group G. The irreducible unitary representation $(\pi, \varkappa_\pi)$ is said to be square-integrable if there exists a non zero vector $v \in \varkappa_\pi$ such that

$$\int_G \left|(v|\pi(g)v)_{\varkappa_\pi}\right|^2 \, dg < \infty. \qquad (6)$$

Such vector is called **admissible.** Furthermore, in the case the representation $(\pi, \varkappa_\pi)$ is square-integrable, Duflo-Moore in [6] found a densely defined and positive self-adjoint unique operator given by

$$K_\pi\colon \varkappa_\pi \to \varkappa_\pi. \qquad (7)$$

which satisfies two following conditions:

1. The neccessary and sufficient conditions for v to be admissible are v in domain K_π.
2. If vectors $f_1, f_3 \in \varkappa_\pi$ and $f_2, f_4 \in \mathrm{Dom}\, K_\pi$, then we have

$$\int_G (f_1|\pi(x)f_2)_{\varkappa_\pi}(\pi(x)f_4|f_3)_{\varkappa_\pi} \, dx = (f_1|f_3)_{\varkappa_\pi}\big(K_\pi(f_4)\big|K_\pi(f_2)\big)_{\varkappa_\pi}. \qquad (8)$$

Let $\mathfrak{g}$ be a 4-dimensional standard filiform Lie algebra with basis $\Delta := \{x_1, x_2, x_3, x_3\}$ whose Lie group is G. The non zero brackets of $\mathfrak{g}$ are given by

$$[x_1, x_2] = x_3, [x_1, x_3] = x_4 \qquad (9)$$

We mention here that in previous work in [2] and [10], we found the representation of G is realized on Hilbert space $\mathrm{L}^2(\mathbb{R})$ as follows:

$$\pi_{\delta,\beta}(g)f(x) := e^{2\pi i\left(\beta b+\delta d+\delta xc+\left(\frac{1}{2}\right)\delta x^2 b+\left(\frac{1}{2}\right)\delta xab+\left(\frac{1}{2}\right)\delta ac+\left(\frac{1}{6}\right)\delta a^2 b\right)}f(x+a), \quad (10)$$

with $g := g(a,b,c,d) \in G$ and $f \in \mathrm{L}^2(\mathbb{R})$. The representation (1) corresponds to 2-dimensional generic coadjoint orbits

$$\Omega_{\delta,\beta} := \left\{\delta x_4^* + s x_3^* + \left(\beta + \frac{s^2}{2\delta}\right) x_2^* + u x_1^* \quad ; \; s,u \in \mathbb{R}, \delta \neq 0\right\}. \quad (11)$$

Contrary to [10] and [2], in this paper we consider the representations of G on Hilbert space $\mathrm{L}^2(\mathbb{R})$ with respect to the basis $\Delta := \{x_1, x_2, x_3, x_3\}$ of $\mathfrak{g}$. We thought that our computations are simpler than previous results and another reason is to attract Indonesian young researcher to study representation theory of Lie groups. We shall claim two our main proposition as follows:

Proposition 1. *Let G be a Lie group of a 4-dimensional standard filiform Lie algebra $\mathfrak{g}$ with basis $\Delta := \{x_1, x_2, x_3, x_3\}$ whose its non zero brackets is given in* (9)*. Then, the irreducible unitary representation of G on the Hilbert space $\mathrm{L}^2(\mathbb{R})$ corresponding to coadjoint orbits $\Omega_{\delta,\beta}$ in* (11) *can be written as follows:*

$$\begin{aligned} \pi_\mu(e^{ax_1})f(x) &= \varphi(x+a), \\ \pi_\mu(e^{bx_2})f(x) &= e^{2\pi i\left(\beta b+\left(\frac{1}{2}\right)\delta x^2 b\right)}\varphi(x), \\ \pi_\mu(e^{cx_3})f(x) &= e^{2\pi i\delta cx}\varphi(x), \\ \pi_\mu(e^{dx_4})f(x) &= e^{2\pi i\delta d}\varphi(x), \end{aligned} \quad (12)$$

where $\mu := \beta x_2^* + \delta x_4^* \in \Omega_{\delta,\beta} \subset \mathfrak{g}^*$ and $\varphi \in \mathrm{L}^2(\mathbb{R})$.

Furthermore, we investigate square-integrability of π_μ of G and we have.

Proposition 2. *The irreducible unitary representation π_μ of G as written in eqs.* (12) *is square-integrable and its Duflo-Moore operator is a multiple of identity.*

METHODS

The method of this research is based on literature survey, specially from [10]. We give another approach in construction of irreducible unitary representation of a Lie group G whose Lie algebra is 4-dimensional standard filiform. We compute it with respect to basis of $\mathfrak{g}$. Therefore, the computations are more simpler than previous work.

RESULTS AND DISCUSSION

In this section, we shall prove our main propositions as already mentioned in introduction

Proposition 1. *Let G be a Lie group of a 4-dimensional standard filiform Lie algebra $\mathfrak{g}$ with basis $\Delta := \{x_1, x_2, x_3, x_4\}$ whose its non zero brackets is given in* (9)*. Then, the irreducible* *unitary representation of G on the Hilbert space $\mathrm{L}^2(\mathbb{R})$ corresponding to coadjoint orbits $\Omega_{\delta,\beta}$ in (12) can be written as follows:*

$$\pi_\mu(e^{ax_1})f(x) = \varphi(x+a),$$
$$\pi_\mu(e^{bx_2})f(x) = e^{2\pi i\left(\beta b+\left(\frac{1}{2}\right)\delta x^2 b\right)}\varphi(x),$$
$$\pi_\mu(e^{cx_3})f(x) = e^{2\pi i\delta cx}\varphi(x),$$
$$\pi_\mu(e^{dx_4})f(x) = e^{2\pi i\delta d}\varphi(x),$$

where $\mu \coloneqq \beta x_2^* + \delta x_4^* \in \Omega_{\delta,\beta} \subset \mathfrak{g}^*$ and $\varphi \in \mathrm{L}^2(\mathbb{R})$.

Proof.

Let $\Delta^* \coloneqq \{x_1^*, x_2^*, x_3^*, x_4^*\}$ be a basis for $\mathfrak{g}^*$. The detail computations for coadjoint orbits of G can be found in ([10], p. 33—34) or ([2], p. 77—80). Let $f = \alpha x_1^* + \beta x_2^* + \gamma x_3^* + \delta x_4^* \in \mathfrak{g}^*$. We recall the result of 2-dimensional coadjoint orbits as follows:

$$\Omega_{\delta,\beta} \coloneqq \left\{\delta x_4^* + s x_3^* + \left(\beta + \frac{s^2}{2\delta}\right)x_2^* + u x_1^* \quad ; \; s,u \in \mathbb{R}, \delta \neq 0\right\}.$$

To obtain the irreducible unitary representations of G corresponding to generic coadjoint orbits $\Omega_{\delta,\beta}$, we consider the following steps

Step 1. We choose a point $f_{\beta,\delta} \coloneqq \beta x_2^* + \delta x_4^* \in \Omega_{\delta,\beta}$.

Step 2. We determine a subalgebra $\Upsilon \coloneqq \mathrm{Span}\{x_2, x_3, x_4\}$ of $\mathfrak{g}$. In this case Υ is commutative since $[x_i, x_j] = 0$ for $1 \le i,j \le 3$. Furthermore, since Υ has maximal dimension i.e. $\mathrm{Codim}\,\Upsilon = \frac{1}{2}\dim\,\Omega_{\delta,\beta} = \frac{1}{2}.2 = 1$. On the other hand, Υ is subordinate to $f_{\beta,\delta}$ since $\langle f_{\beta,\delta}, [\Upsilon,\Upsilon]\rangle = 0$. Thus, Υ is a polarization of $\mathfrak{g}$.

Step 3. Let $\mathrm{H} \coloneqq \exp\Upsilon$ be subgroup and let ψ_f be a 1-dimensional irreducible unitary representations of H given by

$$\psi_{f_{\beta,\delta}}(\exp(ax_1 + bx_2 + cx_3 + dx_4) = e^{2\pi i\langle f_{\beta,\delta},\; X\rangle} = e^{2\pi i(\beta b+\delta d)},$$
$$(f_{\beta,\delta} \in \Omega_{\delta,\beta},\; X = ax_1 + bx_2 + cx_3 + dx_4 \in \mathfrak{g}).$$

Step 4. Identifying the coset space G/H with $\mathbb{R}$ by

$$\mathbb{R} \ni x \mapsto \mathrm{H}\exp(ax_1) \in G/\mathrm{H},$$

we have a section given by

$$s: G/\mathrm{H} \cong \mathbb{R} \ni x \mapsto \exp xx_1 \in G.$$

Step 5. We now solve the master equation with respect to (w.r.t) basis $\Delta \coloneqq \{x_1, x_2, x_3, x_4\}$.

a. W.r.t e^{ax_1}

$$\exp xx_1 \exp ax_1 = \exp(x+a)x_1 \,.$$

b. W.r.t e^{bx_2}

$$\begin{aligned}\exp xx_1 \exp bx_2 &= \exp\left(e^{\operatorname{ad} xx_1}. ax_2\right) \exp xx_1 ,\\ &= \exp(bx_2 + bxx_3 + \left(\tfrac{1}{2}\right) x^2 bx_4) \exp xx_1.\end{aligned}$$

c. W.r.t e^{cx_3}

$$\begin{aligned}\exp xx_1 \exp cx_3 &= \exp\left(e^{\operatorname{ad} xx_1}. cx_3\right) \exp xx_1 ,\\ &= \exp(cx_3 + cxx_4) \exp xx_1.\end{aligned}$$

d. W.r.t e^{dx_4}

$$\begin{aligned}\exp xx_1 \exp dx_4 &= \exp\left(e^{\operatorname{ad} xx_1}. dx_4\right) \exp xx_1 ,\\ &= \exp(dx_4) \exp xx_1.\end{aligned}$$

Using induced representation on 1-dimensional irreducible unitary representations ψ_f of H, we obtain the explicit formulas for ireducible unitary representations of G realized on Hilbert space $\mathrm{L}^2(\mathbb{R})$ written as follows.

$$\begin{gathered}\pi_\mu(e^{ax_1})f(x) = \varphi(x+a),\\ \pi_\mu\left(e^{bx_2}\right)f(x) = e^{2\pi i\left(\beta b+\left(\frac{1}{2}\right)\delta x^2 b\right)}\varphi(x),\\ \pi_\mu(e^{cx_3})f(x) = e^{2\pi i\delta cx}\varphi(x),\\ \pi_\mu\left(e^{dx_4}\right)f(x) = e^{2\pi i\delta d}\varphi(x),\end{gathered}$$

where $\mu \coloneqq \beta x_2^* + \delta x_4^* \in \Omega_{\delta,\beta} \subset \mathrm{g}^*$ and $\varphi \in \mathrm{L}^2(\mathbb{R})$.

■

Proposition 2. *The irreducible unitary representation* π_μ *of* G *as written in eqs.* (12) *is square-integrable and its Duflo-Moore operator is a multiple of identity.*

Proof.

Let φ_1, φ_2 be elements of $\mathrm{L}^2(\mathbb{R})$. For these functions we shall compute the integral

$$\int_G \left|(\varphi_1|\pi(g)\varphi_2)_{\mathrm{L}^2(\mathbb{R})}\right|^2 \, dg \quad .$$

Let us put $g = g'e^{ax_1}$ where $g' = e^{px_4}e^{c_3}e^{bx_2}$. We have

$$(\varphi_1|\pi(g)\varphi_2)_{\mathrm{L}^2(\mathbb{R})} = \int_{\mathbb{R}} \varphi_1(x)e^{2\pi i\left(\delta p+\delta cx+\beta b+\left(\frac{1}{2}\right)\delta x^2 b\right)}\pi(e^{ax_1})\varphi(x) \quad dx$$

By Plancherel's Theorem and Fubini's Theorem of integral we have

$$\int_{\mathbb{R}^4} \left|(\varphi_1|\pi(g)\varphi_2)_{\mathrm{L}^2(\mathbb{R})}\right|^2 \, dpdcdbda = \int_{\mathbb{R}} |\varphi_1(x)|^2 \{\int_{\mathbb{R}} |\varphi_2(x+a)|^2 \, da\} dx$$

$$= \int_{\mathbb{R}} |\varphi_1(x)|^2 \, dx \int_{\mathbb{R}} |\varphi_2(a')|^2 \, da'$$

(we put $a' = x + a$)

$$= \|\varphi_1\|^2_{\mathrm{L}^2(\mathbb{R})} \|\varphi_2\|^2_{\mathrm{L}^2(\mathbb{R})}.$$

The latter equation gives information that π_μ is a square-integrable representation. Therefore, its Duflo-Moore operator is equal to scalar multiple of identity which scalar equals one.

■

We mention here some results in [8] regarding unimodular groups as follows.

1. If G is unimodular then the Duflo-Moore operator K_π in (8) is the scalar multiple of identity. Therefore, the equation (8) is of the form

$$\int_G (f_1|\pi(x)f_2)_{\varkappa_\pi}(\pi(x)f_4|f_3)_{\varkappa_\pi} \, dx = \lambda(f_1|f_3)_{\varkappa_\pi}(f_4|f_2)_{\varkappa_\pi}. \qquad (13)$$

2. If G is unimodular then all vectors in $\varkappa_\pi$ are admissible.

In fact, since $\mathrm{tr} \circ \mathrm{ad} = 0$ then the Lie group G of 4-dimensional standard filiform Lie algebra $\mathfrak{g}$ is unimodular. Therefore, the computations is immediately true as desired. Namely, the irreducible unitary representation π_μ is square-integrable and its Duflo-Moore operator is equal to the scalar λ multiple with $\lambda = 1$.

CONCLUSION

We conclude that irreducible unitary representation π_μ of Lie group of 4-dimensional filiform Lie algebra with respect to its basis is obtained. Furthermore, this representation π_μ is square-integrable and we obtain the Duflo-Moore operator is a scalar multiple of identity and its scalar equals one.

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